Now we have

$$6^\lambda \Sigma N(d)N(m/d) = 6^\lambda \sum_{r_i=0}^{\alpha_i} (\alpha_1 - r_1 + 1)(\alpha_2 - r_2 + 1) \cdots (\alpha_\lambda - r_\lambda + 1)(r_1 + 1)(r_2 + 1) \cdots (r_\lambda + 1)$$

$$= 6^\lambda \sum_{r_i=0}^{\alpha_i} (\alpha_1 - r_1 + 1)(r_1 + 1) \cdots (\alpha_\lambda - r_\lambda + 1)(r_\lambda + 1)$$

$$= 6 \sum_{r_1=0}^{\alpha_1} [\alpha_1(r_1 + 1) - (r_1^2 - 1)] \cdot 6 \sum_{r_2=0}^{\alpha_2} [\alpha_2(r_2 + 1) - (r_2^2 - 1)] \cdots 6 \sum_{r_\lambda=0}^{\alpha_\lambda} [\alpha_\lambda(r_\lambda + 1) - (r_\lambda^2 - 1)]. \tag{2}$$

The indicated summations can be readily performed. We have

$$\sum_{r_i=0}^{\alpha_i} (r_i + 1) = \frac{\alpha_i(\alpha_i + 1)}{2} + \alpha_i + 1 = \frac{(\alpha_i + 1)(\alpha_i + 2)}{2}$$

and

$$\sum_{r_i=0}^{\alpha_i} (r_i^2 - 1) = \frac{\alpha_i(\alpha_i + 1)(2\alpha_i + 1)}{6} - \alpha_i - 1 = \frac{(\alpha_i + 1)(\alpha_i + 2)(2\alpha_i - 3)}{6}.$$

Hence

$$6 \sum_{r_i=0}^{\alpha_i} [\alpha_i(r_i + 1) - (r_i^2 - 1)] = 6\left[\alpha_i \frac{(\alpha_i + 1)(\alpha_i + 2)}{2} - \frac{(\alpha_i + 1)(\alpha_i + 2)(2\alpha_i - 3)}{6}\right]$$
$$= (\alpha_i + 1)(\alpha_i + 2)(\alpha_i + 3).$$

Substituting for these sums in the last member of (2) and making use of (1), we have

$$6^\lambda \Sigma N(d)N(m/d) = (\alpha_1 + 1)(\alpha_1 + 2)(\alpha_1 + 3)(\alpha_2 + 1)(\alpha_2 + 2)(\alpha_2 + 3) \cdots (\alpha_\lambda + 1)(\alpha_\lambda + 2)(\alpha_\lambda + 3)$$
$$= N(m)N(Pm)N(P^2m).$$

197. Proposed by E. T. BELL, Seattle, Washington.

Show that in the expansion of

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,$$

where p is a prime, the coefficients of the various powers of z are divisible by p.
[Eisenstein, *Crelle*, t. 27, p. 282.]

SOLUTION BY B. F. YANNEY, University of Wooster.

The given expression is evidently equal to $(1 - z^p)/(1 - z)^p - 1$, which may be put in the form $(1 - z^p)(1 - z)^{-p} - 1$. Expanding the second factor of the first term and noticing that, since p is prime, the coefficient of each term

in the expansion except that of the 1st, the $(p+1)$th, the $(2p+1)$th, and so on to the $(np+1)$th, and so on, is divisible by p, we obtain, omitting the terms with the divisible coefficients,

$$(1-z^p)\left(1+\frac{p(p+1)\cdots(2p-1)}{p!}z^p+\frac{p(p+1)\cdots(3p-1)}{(2p)!}z^{2p}+\cdots\right.$$
$$\left.+\frac{p(p+1)\cdots((n+1)p-1)}{(np)!}\cdots\right)-1.$$

Performing the indicated operations, we get

$$\left(\frac{p(p+1)\cdots(2p-1)}{p!}-1\right)z^p+\left(\frac{p(p+1)\cdots(3p-1)}{(2p)!}-\frac{p(p+1)\cdots(2p-1)}{p!}\right)z^{2p}$$
$$+\cdots+\left(\frac{p(p+1)\cdots((n+1)p-1)}{(np)!}-\frac{p(p+1)\cdots(np-1)}{((n-1)p)!}\right)z^{np}+\cdots.$$

The first coefficient may be written $[(p+1)(p+2)\cdots(p+p-1)-(p-1)!]/(p-1)!$, which is equal to $[pA+(p-1)!-(p-1)!]/(p-1)!$, where A is a polynomial in p. This expression, which is equal to $pA/(p-1)!$, and is an integer, as are all the coefficients, is plainly divisible by p, since the denominator does not contain p as a factor.

The coefficient of the general term may be changed in form to

$$\frac{p(p+1)\cdots(np-1)}{((n-1)p)!}\left(\frac{np(np+1)\cdots(np+p-1)}{[(n-1)p+1][(n-1)p+2]\cdots[np]}-1\right).$$

The first factor of this is an integer; hence so is the second, which latter can be expressed in the form

$$\frac{(np+1)(np+2)\cdots(np+p-1)-[(n-1)p+1][(n-1)p+2]\cdots[(n-1)p+p-1]}{[(n-1)p+1][(n-1)p+2]\cdots[(n-1)p+p-1]},$$

which equals

$$\frac{(npB+(p-1)!)-((n-1)pC+(p-1)!)}{(n-1)pC+(p-1)!},$$

where B and C are polynomials in np and $(n-1)p$, respectively. Since the denominator does not contain p as a factor, but the numerator does, the expression is divisible by p, which completes the proof.

Also solved by ELMER SCHUYLER, H. C. FEEMSTER and the PROPOSER.